ON THE GRAPHIC SOLUTION OF DYNAMICAL PROBLEMS

BY

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1. In a paper published under the above title in the *Philosophical Magazine and Journal of Science* for 1892, Lord Kelvin described a graphic method for the solution of dynamical problems. We will quote from his paper the following account of his method.

«In dynamical problems regarding the motion of a single particle in a plane the following plan is given for drawing any possible path under the influence of a force of which the potential is given for every point of the plane. Suppose, for example, it is required to find the path of a particle projected, with any given velocity, in any given direction through any given point P_0 (see fig. 1)».

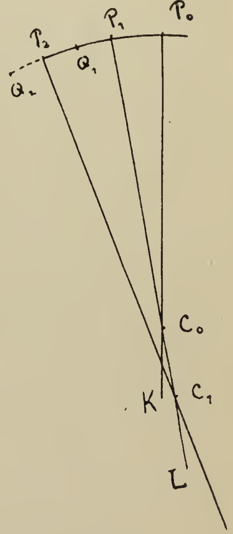

Fig. 1.

«Calculate the normal component force at this point, and divide the square of the velocity by this value, to find the radius of curvature of the path at that point. Taking this radius on the compasses, find the centre of curvature C_0 in the line P_0K, perpendicular to the given direction through P_0, and describe a small arc, $P_0P_1Q_1$, making P_1Q_1 equal to about half the length intended for the second arc. Calculate the altered velocity for the position Q_1, according to the potential law, and, as before for P_0, calculate a fresh radius of curvature for Q_1 by finding the normal component force for the altered direction of normal, and for the velocity corresponding to the position of Q_1. With this radius, find the position of the centre of curvature, C_1, in P_1C_0L, the line of the radius through P_1. With this centre of curvature and the fresh radius of curvature, describe an arc $P_1P_2Q_2$, making P_2Q_2 equal to about half the length intended for the third arc; calculate radius of curvature for position Q_2, draw an arc $P_2P_3Q_3$, and continue the procedure».

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The practical utility of this method is dependent upon the ease or difficulty with which the radius of curvature can be calculated. If the potential on which the force depends is a somewhat complicated function of the coördinates of the point, the method immediately becomes so troublesome to use, that recourse should rather be had to one of the many numerical methods of integration for the calculation of the path.

We intend, however, to demonstrate *an extremely simple, approximate construction of the radius of curvature by the aid of a number of equipotential lines, whereby the above difficulty is avoided.*

Let the differential equations for the motion of the point be

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{\partial U}{\partial x} \\ \frac{d^2y}{dt^2} &= \frac{\partial U}{\partial y} \\ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 2U \end{aligned}\right\} \qquad (I)$$

where U is a given function of x and y. Let us further assume that we have drawn in advance a number of equipotential lines,

$$U = 0, \quad U = \frac{1}{n}, \quad U = \frac{2}{n}, \quad U = \frac{3}{n}, \quad \ldots\ldots \quad U = \frac{p}{n}, \quad \ldots\ldots$$

Let $A\,M\,B$ be a part of the path, and M its point of intersection with the equipotential line $U = \frac{p}{n}$. Further let $M\,N$ be the normal, with direction towards increasing Q, i. e. towards the centre of curvature of the path. *If we designate as* D *the part of the normal between the two nearest equipotential lines,* $U = \frac{p-1}{n}$ *and* $U = \frac{p+1}{n}$, *we have for the radius of curvature* ϱ *in the point* M, *the approximate formula*

$$\varrho = pD.$$

By the aid of this formula, the radius of curvature may be easily and quickly constructed, when the equipotential lines have been drawn *once for all.* The construction of the path of the point by the aid of the radii of curvature thereby becomes very simple and easy of accomplishment. We will return later to the question of the possibility of modifying the method when the radius of curvature is very large, or when the equipotential lines lie far from one another.

We will first prove the correctness of the approximate formula $\varrho = pD$. If we designate the projection on the normal $M\,N$ of the force acting on the point M, as K_n, and the velocity of the point as v, we have, as we know, the mass of the point being equal to 1,

$$K_n = \frac{v^2}{\varrho}$$

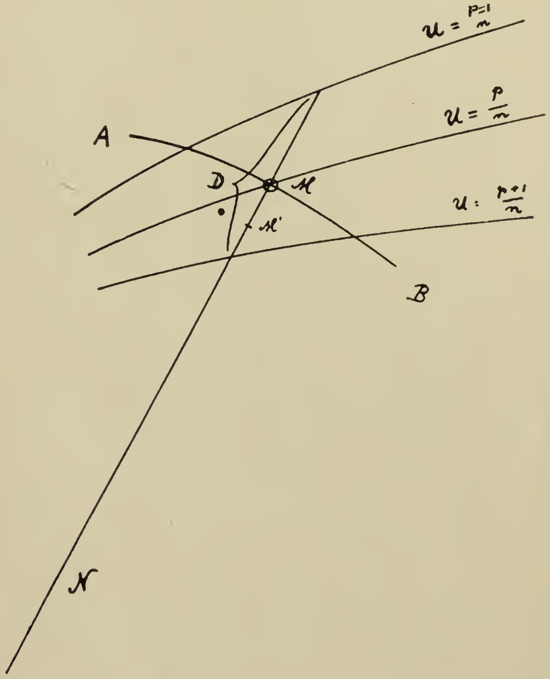

Fig. 2.

According to system I, $v^2 = 2U$, and $K_n = \frac{\partial U}{\partial N}$, where $\frac{\partial U}{\partial N}$ indicates the derivative of U along the normal $M\,N$. This gives

$$\frac{1}{\varrho} = \frac{1}{2U}\frac{\partial U}{\partial N} \tag{II}$$

Let us now consider an assumed point M' on the normal, and call its distance from a fixed point on the latter ξ, ξ to be reckoned positive in the direction $M\,N$. If M' moves, U will be a function of ξ, and if this function be developed in series, we easily obtain the desired approximate formula, $\varrho = pD$. It was in this way, moreover, that I first demonstrated this formula.

We attain our end most rapidly, however, in accordance with Professor Runge's suggestion, by regarding ξ as a function of U, and considering the corresponding curve,

$$\xi = f(U)$$

where U is an abscissa and ξ an ordinate[1]. Let A, B and C be the three points whose abscissæ are $\frac{p-1}{n}$, $\frac{p}{n}$ and $\frac{p+1}{n}$, and let AA' be parallel with the axis of abscissæ; then $AA' = \frac{2}{n}$, and $A'C = D$.

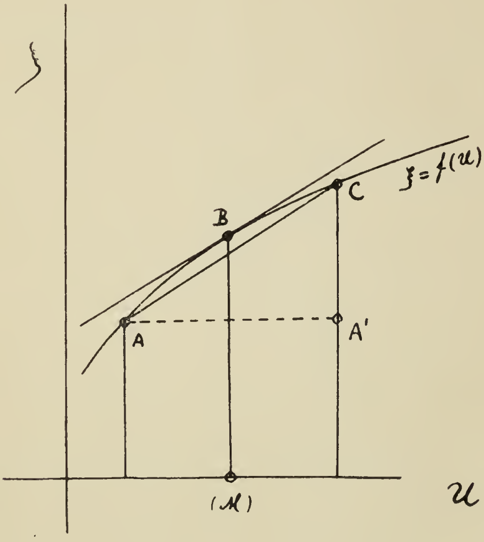

Fig. 3.

Thus if, instead of the direction of the tangent in the point B, we take as approximate direction AC, we obtain

$$\tan \beta = \frac{n}{2} D,$$

β being the angle made by the tangent with the axis of abscissæ.

On the other hand, $\tan \beta = f'(U)$, where the value of U is $\frac{p}{n}$. Now since

$$f'(U) = \frac{1}{\frac{\partial U}{\partial N}} = \frac{\varrho}{2U},$$

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[1] I am indebted to Professor Runge for information respecting this method of procedure. At the beginning of March, 1907, I had sent a description of the method with proofs to Professor Mittag Leffler, asking if it were known. He sent it to Professor Runge in Göttingen, who drew attention to the above simpler proof.

we also obtain for tan β the value $\frac{\varrho n}{2p}$. If we compare these two expressions for tan β, we obtain

$$\varrho = pD \qquad\qquad Q. E. D.$$

It is easy to find an expression for the accuracy attained, for, according to Taylor's formula, we have

$$f(U_0 + \Delta U) = f(U_0) + f'(U_0)\,\Delta U + \frac{f''(U_0)}{1 \times 2}(\Delta U)^2 + \frac{f'''(U_0 + \theta\,.\,\Delta U)}{1 \times 2 \times 3}(\Delta U)^3$$

and

$$f(U_0 - \Delta U) = f(U_0) - f'(U_0)\,\Delta U + \frac{f''(U_0)}{1 \times 2}(\Delta U)^2 - \frac{f'''(U_0 - \theta'\,.\,\Delta U)}{1 \times 2 \times 3}(\Delta U)^3,$$

where θ and θ' are numbers between 0 and 1. From this we get

$$f(U_0 + \Delta U) - f(U_0 - \Delta U) = 2f'(U_0)\,\Delta U + \tfrac{1}{6}[f'''(U_0 + \theta.\Delta U) + f'''(U_0 - \theta'.\Delta U)](\Delta U)^3.$$

If we now choose $U_0 = \frac{p}{n}$ and $\Delta U = \frac{1}{n}$, we obtain

$$D = \frac{\varrho}{p} + \frac{1}{6n^3}\left[f'''\left(\frac{p+\theta}{n}\right) + f'''\left(\frac{p-\theta'}{n}\right)\right],$$

and thus

$$\varrho = pD + \varepsilon,$$

where we have put

$$\varepsilon = -\frac{p}{6n^3}\left[f'''\left(\frac{p+\theta}{n}\right) + f'''\left(\frac{p-\theta'}{n}\right)\right].$$

Now if $f'''(U)$ in the interval $\frac{p-1}{n} \leqq U \leqq \frac{p+1}{n}$, lies between $-m$ and $+m$, then

$$-\frac{p}{3n^3}m < \varepsilon < \frac{p}{3n^3}m\,.$$

In order to find m, $f'''(U)$ may be expressed by $\frac{dU}{d\xi}$, $\frac{d^2U}{d\xi^2}$ and $\frac{d^3U}{d\xi^3}$. As $\xi = f(U)$, we have

$$f'''(U) = \frac{3\left(\frac{d^2U}{d\xi^2}\right)^2 - \frac{dU}{d\xi}\,\frac{d^3U}{d\xi^3}}{\left(\frac{dU}{d\xi}\right)^5}$$

Thus the first three derivatives of the function U along the normal MN may be formed, and $f'''(U)$ may then be found by substitution in the above formula.

We see that the error ε diminishes very rapidly with increasing n.

2. The above method was worked out for the purpose of simplifying the study of the integral curves of the system

$$\left.\begin{aligned} \frac{d^2R}{ds^2} &= \tfrac{1}{2}\frac{\partial Q}{\partial R} \\ \frac{d^2z}{ds^2} &= \tfrac{1}{2}\frac{\partial Q}{\partial z} \\ \left(\frac{dR}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 &= Q \end{aligned}\right\} \qquad \text{(III)}$$

where Q is the function

$$Q = 1 - \left[\frac{2\gamma}{R} + \frac{R}{(R^2 + z^2)^{\frac{3}{2}}}\right]^2$$

and γ is constant.

One is led to this system, when it is a question of calculating the trajectories of electric corpuscles in space under the influence of terrestrial magnetism[1].

One of my assistants, Mr. E. Steenstrup, engineer, has constructed under my guidance, after the above method, more than 50 paths of system (III), answering to various values of the constant γ. The practical utility of the method has thereby been sufficiently tested. The results of these constructions will be fully described in Professor Birkeland's report of the Norwegian auroral expedition of 1902—3, in the section that treats of the mathematical theory of auroras and magnetic perturbations.

3. We will here briefly relate our experience with regard to the employment of our graphic method of finding the paths defined by system (I).

First a number of equipotential curves,

$$U = 0, \quad U = \frac{1}{n}, \quad U = \frac{2}{n}, \quad U = \frac{3}{n}, \; \ldots\ldots$$

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[1] See my paper: „Sur les trajectoires des corpuscules électrisés dans l'espace sous l'action du magnétisme terrestre, avec application aux aurores boréales". Archives des Sciences Physiques et Naturelles; July—October, 1907. Geneva.

are calculated, and the curves drawn upon paper. n is assumed so large that the curves on the diagram are not too far distant from one another; in some places it will generally be necessary to assume larger values for n than at other places. For the sake of clearness, it is recommended that all curves $U = \frac{p}{n}$ be dotted lines for p odd, and continuous lines for p even, and that the odd numbers, 1, 3, 5, 7, etc., be written in the spaces between the continuous curves, from $U = 0$ onwards, as shown in figure 4.

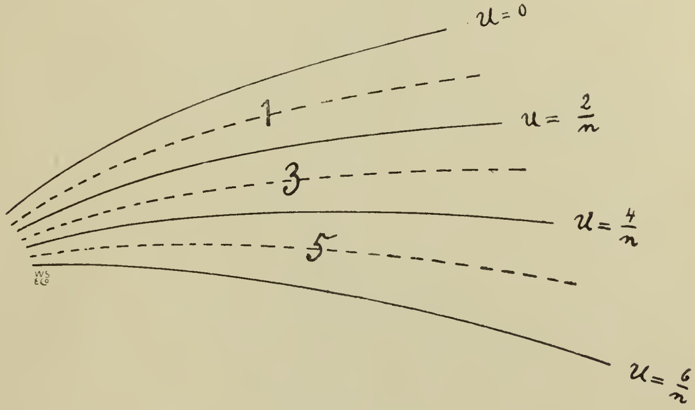

Fig. 4.

If the point M of the path falls upon one of the dotted curves, the radius of curvature then equals the portion D of the normal between the two nearest continuous curves multiplied by the number of the interval. If M falls upon one of the continuous curves, we have to multiply D by the even number lying between the numbers of the adjacent intervals.

The graphic construction of the path may now be accomplished in the manner given by Lord Kelvin; but in addition to compasses, it is an advantage to use a narrow tape divided into millimetres, by which the centre of curvature can be quickly marked. By means of pins, this tape is attached to, and detached from, the points in the path and their centres of curvature alternately, as the construction progresses. The product pD may be found by the aid of a sliding-rule.

More accurate results are obtained by substituting transparent gelatine-paper, cross-ruled, for the tape.

Before proceeding to a treatment of the cases in which the method must be modified, we will consider the connection between the radius of curvature ϱ of an assumed path through the point M (see fig. 5), and the radius of curvature ϱ_0 of a path through the same point, tangent to the equipotential line through that point. If we call the projection of the force K along $M\,N$, as before, K_n, we have

$$K_n = K \cos \psi .$$

Now since

$$K = \frac{v^2}{\varrho_0} = \frac{2U}{\varrho_0},$$

and

$$K_n = \frac{v^2}{\varrho} = \frac{2U}{\varrho},$$

we obtain

$$\varrho = \frac{\varrho_0}{\cos \psi},$$

whence we have the construction of ϱ shown in fig. 5, when ϱ_0 and the angle ψ are given.

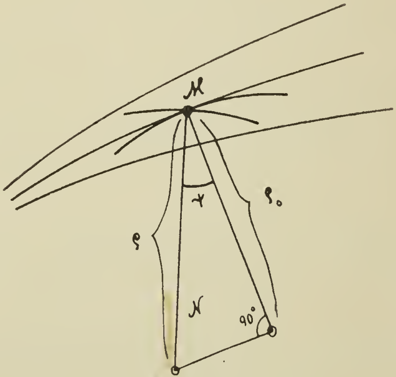

Fig. 5.

This construction is to be recommended, *when there is a suspicion that the formula* $\varrho = pD$ *does not give exact results, owing to the fact that the path is becoming perpendicular to the equipotential curves, or because the course of these curves is not sufficiently regular in the vicinity of the point M.* By the aid of the already-mentioned cross-rule gelatine-paper, the factor $\frac{1}{\cos \psi}$ is quickly found as follows.

A cross of axes is made on the gelatine-paper, consisting of two axes at right angles to one another, one an axis of abscissæ $x'x''$, and the other an axis of ordinates $y'y''$. A needle is then put through the intersection of the axes and the point M in the path, and the gelatine-paper is turned in such a manner that the axis $x'x''$ is tangent to the path. Along the normal $M\,N$, a portion, $M\,N'$, is then marked off, of a given length a, and N' is marked upon the paper by making a hole with a needle through the gelatine. The gelatine is then turned about M until the axis $x'x''$ is tangent to the equipotential curve through M, and the distance, b, from the point marked on the paper to the axis $x'x''$, is read off. Then $\cos\psi = \frac{b}{a}$ and thus

$$\varrho = \frac{a}{b}\,\varrho_0\,,$$

whereupon ϱ is quickly calculated with a sliding-rule.

In the case of ϱ *being so large that the centre of curvature falls outside the paper,* the following method may be employed.

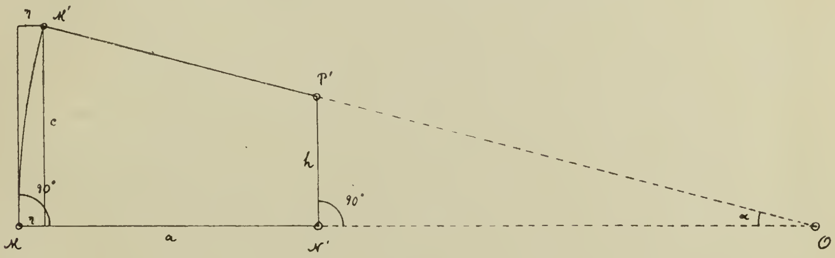

Fig. 6.

In this figure, where MM' is an arc of a circle with radius ϱ and centre in O (centre of curvature), we have

$$\eta = \varrho\,(1 - \cos\alpha)$$

$$c = \varrho\sin\alpha$$

consequently, with sufficient accuracy,

$$\eta = \frac{c^2}{2\varrho}\,,$$

$$c - h = \frac{a}{\varrho}\,c$$

Here c is assumed, as also a. The position of M' is thereby determined, η and h being easily calculated with a sliding-rule. As also the position of P, is known, we have the direction of the radius of curvature in the point M'.

If the gelatine-paper is used, it is turned about M so that the one axis falls along MN'. By the aid of the cross-rule paper, the points M' and P' may then easily be marked.

This is the description, in its principal details, of the graphic construction of the path.

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